

Introduction in Quantum Information Theory.  
script

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# Chapter 1

## Introduction.

One can learn classic informatics without learning how the computers are built. The knowledge of electrodynamics or electronics is normally not required. The hardware plays the role of abstraction layer and simulates a simple framework of binary logic used in the theory and in the application programming. The quantum information theory on the contrary uses rather contra-intuitive concept of quantum state and measurement process. This makes learning the quantum informatics complicate: the fundamentals of the quantum mechanics are required even for the basic concepts of the theory. The student is intentionally exposed to the physics of the hardware or at least to the fundamental quantum concepts.

In this script we prefer to face this challenge from very beginning and introduce the concepts of the quantum mechanics and the required mathematical tools first, before we start with the information theory. We avoid any discussion of physical motivation and of the historical perspective, which are domain of physicists, and choose an axiomatic approach, which should deliver the formal framework and build the ground for everything, what follows.

# Chapter 2

## Axioms of the quantum mechanics.

Physics typically considers entity called 'physical system' and tries to find a mathematical description, which would allow to predict the evolution of the system in time. An observer can make experiments (observations) and measure different physical properties (observables) of the system. Target of the theory is to predict the result of such experiments. If the deterministic prediction is not possible, the probability distribution of possible values would be the reduced target.

We start with axioms of the quantum theory. No attempt is made to explain why the axioms are so as they are.

These notes are compiled from few classic sources listed in bibliography at the end of the script. The books listed there cover the subject in much more details.

### 2.1 Mathematical notation

To formulate axioms we will need few mathematical concepts introduced in this section. Vector space  $V$  over the number field  $K$ , called  $K$ -vector space, is a set of objects called 'vectors'  $V = \{|v\rangle\}$  together with two operations called addition and multiplication by a number:

$$+ : V \times V \rightarrow V \tag{2.1}$$

$$* : K \times V \rightarrow V \tag{2.2}$$

$$\tag{2.3}$$

Note: we distinguish vector from numbers by putting them into strange brackets like this:  $|v\rangle$ . After Dirac this notation is called KET from bra-cket.

Addition should be commutative, both operations should be associative and distributive law must hold:  $\alpha(|v\rangle + |w\rangle) = \alpha|v\rangle + \alpha|w\rangle$ . Additionally the existence of the zero vector  $|0\rangle$  with properties  $|v\rangle + |0\rangle = |v\rangle$ ,  $0|v\rangle = |0\rangle$  and  $\alpha|0\rangle = |0\rangle$  for all vectors  $|v\rangle \in V$  and all numbers  $\alpha \in K$  is required.

In the quantum theory the field  $K$  is the field  $\mathbb{C}$  of complex numbers and the state vector space is a  $\mathbb{C}$ -vector space. One can add the third operation called scalar product:

$$\langle v|w\rangle : V \times V \rightarrow K \tag{2.4}$$

Note: we write co-vector (also called dual vector) into reverse bracket like this:  $\langle v|$ . After Dirac this notation is called BRA from brac-ket.

For the  $\mathbb{C}$ -vector space the scalar product is typically required to satisfy the following conditions for all vectors  $v, v' \in V, w, w' \in W$  and  $\alpha \in K$ :

$$\langle v|v \rangle = r; r \in \mathbb{R}, r \geq 0 \quad (2.5)$$

$$\langle v|v \rangle = 0 \Leftrightarrow v = |0\rangle \quad (2.6)$$

$$\langle v|w \rangle = \langle w|v \rangle^* \quad (2.7)$$

$$\langle v + v'|w \rangle = \langle v|w \rangle + \langle v'|w \rangle \quad (2.8)$$

$$\langle v|w + w' \rangle = \langle v|w \rangle + \langle v|w' \rangle \quad (2.9)$$

$$\langle v|\alpha w \rangle = \langle \alpha^* v|w \rangle = \alpha \langle v|w \rangle \quad (2.10)$$

$$|\langle v|w \rangle|^2 \leq \langle v|v \rangle \langle w|w \rangle, \quad (2.11)$$

where  $*$  means complex conjugation. The scalar product of a vector with itself  $\langle v|v \rangle$  is a square of the 'length' of this vector. The (2.11) is called Schwarz inequality.

Vector space with such scalar product is called Hilbert space.

As next we consider a linear operator, that means mapping  $\hat{O} : V \rightarrow V$  with following properties of linearity which hold for each number  $\alpha$  and vectors  $|v\rangle$  and  $|w\rangle$ :

$$\hat{O}(\alpha|v\rangle) = \alpha\hat{O}(|v\rangle) \quad (2.12)$$

$$\hat{O}(|v\rangle + |w\rangle) = \hat{O}(|v\rangle) + \hat{O}(|w\rangle) \quad (2.13)$$

We will simply write  $\hat{O}|v\rangle$  instead of  $\hat{O}(|v\rangle)$ . As with functions, we write  $\hat{A} = \hat{B}$  if  $\hat{A}|v\rangle = \hat{B}|v\rangle$  holds for all vectors  $|v\rangle$ .

For a given operator  $\hat{O}$  it would be interesting to find a new operator  $\hat{O}^\dagger$  with the property  $\langle v|\hat{O}w \rangle = \langle \hat{O}^\dagger v|w \rangle$  holding for all vectors  $|v\rangle$  and  $|w\rangle$ . The operator  $\hat{O}^\dagger$  is called adjoint to operator  $\hat{O}$ . If  $\hat{O} = \hat{O}^\dagger$  the operator  $\hat{O}$  is called self-adjoint or Hermetian.

Vector  $|v\rangle$  is called eigenvector of operator  $\hat{O}$  and number  $\lambda$  is called eigenvalue of operator  $\hat{O}$  corresponding to eigenvector  $|v\rangle$ , if following holds:

$$\hat{O}|v\rangle = \lambda|v\rangle. \quad (2.14)$$

Note that eigenvector remains to be eigenvector (with the same eigenvalue) if multiplied by a number. An Hermetian operator has two important properties: its eigenvalues are real numbers and one can select its eigenvectors to be orthonormal and to constitute basis of the vector space. Recall: the set of vectors is called orthonormal if all of them have the length one and any two of them are orthogonal. In other word:  $\langle v_i|v_j \rangle = \delta_{ij}$ , where  $|v_i\rangle$  and  $|v_j\rangle$  are eigenvectors and  $\delta_{ij}$  is 1 if  $i = j$  and 0 otherwise.

## 2.2 Axiom 1: States of the physical system

How the physical system is represented in the theory?

Axiom 1: State of the physical system is mathematically described as a vector in the Hilbert vector space.

Explanation:

As described above, a vector space is a set of objects called vectors which can be multiplied with numbers (real or complex) and added to each other. As an illustration, one can think about vectors as arrows drawn on the paper: these can also be multiplied by numbers (typically real numbers) and one can add two vectors. The calculation axioms of the vector space are well known from school and can be found in every book or in internet. An important additional operation is the scalar product: two vectors can be

multiplied to produce a complex number. Vector space with defined scalar product is called Hilbert space. The scalar product of the vector with itself can be thought as a square of its 'length'.

The space which describes the physical system can be more complex as the usual two or three dimensional vector space learned in the school. It can have infinitely many dimensions but the axioms and operations remain the same.

It turns out that the length of vector do not play any role in the mathematical description of the physical state. So we can agree to normalize state vectors to the unit length by multiplying them with appropriate real number.

Note: the same physical system can be described with different vector spaces and the same state will then be represented by different vectors. The situation is similar to the question which coordinate system one should use: it depends on the physical system of interest and on the personal preferences.

## 2.3 Axiom 2: Observables

How the experiment is represented in the theory? What do 'experiment', 'measure' and 'observable' mean?

Axiom 2: An observable is mathematically described by linear operator on the vector space of state vectors.

Explanation: Observable is the physical property of interest. One can measure observable by conducting an experiment in which physical system interacts with measure device and changes the state (display) of the device. Examples of observables: velocity, position, temperature etc.

Linear operator on the vector space is a linear mapping which takes one vector as argument and produce a new vector as output. The word 'linear' means that the result is the same whether one applies the operator to the sum of vectors or to each vector separately and adds the vector afterwards. Similarly the result is the same whether one first multiplies the vector with a number and than applies the operator or the other way around. You can think about rotation as a linear operator. Another example is multiplication with a number.

Once the vector space used for description of the physical system is selected, each observables corresponds to particular linear operator in this space.

## 2.4 Axiom 3: Possible results of observation

Which result can produce observation of the given observable?

Axiom 3: An observation result is always an eigenvalue of the operator, which corresponds to the observable in the sense of Axiom 2.

Explanation: If the vector resulting from action of operator equals to the original vector multiplied with a number, then the original vector is called eigenvector and the number is called eigenvalue (corresponding to this eigenvector). For given operator the set of eigenvectors and corresponding eigenvalues is fixed. The Axiom states that observation always results in value (for exmaple velocity value) which is eigenvalue of the 'observation operator' (for example a velocity operator).

As the observation values must be real numbers, the operators corresponding to the oservables must be of such kind, that their eigenvalues are real. Fortunately there is in fact an important class of linear operators which satisfy this condition. Such an operator  $O$  is called self-adjoint or Hermetian and has a property that for any two vectors  $a$  and  $b$  the scalar product  $\langle a, Ob \rangle$  is equal to  $\langle Oa, b \rangle$ . Even more: the eigenvectors of selfajoint operators form a basis of the vector space and any vector can uniquely be written as linear combination of these eigenvectors.

## 2.5 Axiom 4: Probability of the observation results

How to predict the result of observation?

Axiom 4: Assume the system is the particular state  $|v\rangle$  prior to observation. The probability of particular observation result is equal to square module of the scalar product of the state vector  $|v\rangle$  with the eigenvector corresponding to the observation value of interest.

Explanation: Recall that the system is described by vector and that an observation on this system can result only in eigenvalues of the corresponding operator. Each eigenvalue belongs to particular eigenvector. The Axiom states that the probability to measure particular value is the square module of the (complex) scalar product of the state vector and the corresponding eigenvector.

In other words: as the eigenvectors  $\{|\phi_n\rangle : \hat{A}|\phi_n\rangle = \lambda_n |\phi_n\rangle\}$  of the observable operator  $\hat{A}$  form an orthonormal basis of the vector space, one can uniquely project any given state vector  $|\phi\rangle$  on to any given eigenvector  $|\phi_n\rangle$ . The square module of this projection  $p_n = |\langle\phi_n|\phi\rangle|^2$  is the probability to get a measurement result equal to the eigenvalue  $\lambda_n$  of this eigenvector  $|\phi_n\rangle$ .

The expectation value  $\langle A \rangle_\Phi$  of the observables  $A$  is the average of many hypothetical measurements performed on the same state  $|\Phi\rangle$ :

$$\langle A \rangle_\Phi = \sum_n p_n \lambda_n = \sum_n |\langle\phi_n|\Phi\rangle|^2 \lambda_n = \sum_n \langle\Phi|\phi_n\rangle \langle\phi_n|A|\Phi\rangle = \langle\Phi|A|\Phi\rangle \quad (2.15)$$

This formula for the average value of an observable is valid, when the system is in particular state  $\Phi$ , which is in general a superposition of eigenstates of the observable. In many situations it is necessary to generalize (2.15) for a system which state is not known exactly but is given as probability distribution.

Consider a set of possible states  $\{\Phi_n\}$ , which do not need to be mutual exclusive (orthogonal) and let the observable  $A$  be measured many times by selecting a random system from an ensemble composed of systems distributed between these states so, that the state  $\Phi_n$  has a probability  $p_n$ . The probabilities should sum up to one. The average result of the observation is then the weighted sum:

$$\begin{aligned} \langle A \rangle &= \sum_m p_m \langle A \rangle_{\Phi_m} = \sum_m p_m \langle \Phi_m | A | \Phi_m \rangle = \sum_{m,n} p_m \langle \Phi_m | \phi_n \rangle \langle \phi_n | A | \Phi_m \rangle \\ &= \sum_{m,n} \sum_{a,b} p_m c_{ma}^* \langle \phi_a | \phi_n \rangle \langle \phi_n | A | c_{mb} \phi_b \rangle = \sum_{m,n} p_m a_n |c_{nm}|^2, \end{aligned} \quad (2.16)$$

where we take  $|\phi_b\rangle$  to be the eigenvector of  $\hat{A}$  with the eigenvalue  $a_b$ .

To simplify this we introduce the density operator  $\rho = \sum_m p_m |\Phi_m\rangle \langle \Phi_m|$  and write the matrix element  $(\rho A)_{ab}$  as follows:

$$(\rho A)_{ab} = \langle \phi_a | \sum_m p_m \Phi_m \rangle \langle \Phi_m | A | \phi_b \rangle = \sum_m p_m c_{am} c_{bm}^* a_b. \quad (2.17)$$

Comparing this with (2.16) we conclude

$$\langle A \rangle = \text{Tr}(\rho A) = \sum_{m,n} a_n p_m |c_{nm}|^2. \quad (2.18)$$

Note that (2.15) is a particular case of (2.18) when  $\{\Phi_n\} = \{\Phi\}$ .

The (2.18) also helps to see the practical difference between a superposition of the states and a statistical mix of the states in an ensemble. Lets start with two orthonormal eigenvectors  $|+\rangle$  and  $|-\rangle$  of a hypothetical observable  $A$  with the corresponding eigenvalues  $+1$  and  $-1$ . We can build two new orthonormal vectors out of them:

$$\begin{aligned} |0\rangle &= (|+\rangle + |-\rangle) / \sqrt{2} \\ |1\rangle &= (|+\rangle - |-\rangle) / \sqrt{2}. \end{aligned} \tag{2.19}$$

Now consider a system in the state  $|+\rangle$  which is a symmetric superposition of  $|0\rangle$  and  $|1\rangle$  with the probability to find each of them equal to  $\frac{1}{2}$ . The expectation value of  $A$  for this state is by definition  $+1$ . Compare this with the statistical ensamble composed on both states  $|0\rangle$  and  $|1\rangle$  with the same probability of  $\frac{1}{2}$ . The expectation value according to (2.18) is:

$$\begin{aligned} \langle A \rangle &= Tr(\rho A) = \sum_{m,n} a_n p_m |c_{nm}|^2 \\ &= (+1) * \frac{1}{2} * \left| \frac{1}{\sqrt{2}} \right|^2 + (-1) * \frac{1}{2} * \left| \frac{1}{\sqrt{2}} \right|^2 + (+1) * \frac{1}{2} * \left| \frac{1}{\sqrt{2}} \right|^2 + (-1) * \frac{1}{2} * \left| -\frac{1}{\sqrt{2}} \right|^2 = 0. \end{aligned} \tag{2.20}$$

The desity operator, thus, does correctly reflect the difference in these two configurations: in the first case (using the based  $\{|+\rangle, |-\rangle\}$ ) the density matrix has only one non-zero element  $\rho_{11} = 1$ . In the second case the density matrix is diagonal with  $\rho_{ij} = \frac{1}{2}\delta_{ij}$ .

## 2.6 State after observation.

Measurement is a process of interaction of the system with the measurement apparatus and with the observer. Which state has a system after the measurement? Some text books refer to the complexity of such interaction and invoce mysterious collapse of the state vector during the measurement, which results in projection of the vector onto the eigen vector corresponding to the measured value.

This is equivalent to an additional axiom which states, that during the measure yielding result  $\lambda_n$  the state vector  $|\phi\rangle$  collapses to  $|\phi_n\rangle$  multiplied by some not further known phase factor  $e^{i\alpha}$ . The nice consequence of this would be that the measurement performed shortly after the first one would, according to the Axiom 4, certainly produce the same result  $\lambda_n$ .

There are at least two problems with this rule: first, it obviously does not apply to measurements which change the measured property or even destroy the measured object, and second, if the quantum machanics pretends to explain the world, it should also explain what happens during measurement, and not proclame the resulting state as an axiom.

We will revisit this question in more details after discussing Schrodinger equation, which describes the time evaluation of the system. This evolution should be able to explain how the state vector evolves during the process of measurement.

## Chapter 3

# Coordinates, Impuls, Energy

### 3.1 Coordinates

The axioms listed above are quite abstract. How to apply them to the particular physical system? We shall start with a free particle which moves from the left to the right along the  $x$ -axis.

Question: which vector space should be used to describe the particle which can move only in one dimension along the  $x$ -axis?

Answer: The space of complex valued function of coordinate  $x$ :  $\{f(x) : \mathbb{R} \Rightarrow \mathbb{C}\}$  with scalar product defined as follows:

$$\langle f|g \rangle = \int_{-\infty}^{+\infty} f(x)^* g(x) dx. \quad (3.1)$$

It is obviously a vector space (over the field of complex numbers) as one can add functions and multiply with complex numbers.

We could spend more time discussing different possible choices for the state vector space of the single-particle system and some of them could also work, but the choice described above is simple and effective.

Question: which operator should represent the  $x$ -coordinate observable?

Answer: the operator which simply multiply the vector with  $x$  variable:  $\hat{X} : \hat{X}|f\rangle = x|f\rangle = x f(x)$ . Note that  $x f(x)$  is again a function of coordinate  $x$  and so a new valid state vector.

The eigenvectors with eigenvalue  $x_0$  are thus defined by equation  $x f(x) = x_0 f(x)$  satisfied by Dirac's delta function  $\delta(x - x_0)$ .

Consider a particle in the state  $|\xi\rangle = \xi(x)$ . The probability to find this particle at position  $x = x_0$  is, according to axioms, just the square module of projection on the eigenvector  $|\xi_{x_0}\rangle = \delta(x_0)$  with the eigenvalue  $x_0$ :

$$P(x_0) = |\langle \xi | \xi_{x_0} \rangle|^2 = \left| \int_{-\infty}^{+\infty} \xi(x)^* \delta(x_0) dx \right|^2 = |\xi(x_0)|^2. \quad (3.2)$$

The state vector function  $\xi(x)$  is called 'wave function'. The square module of the wave function evaluated on position  $x_0$  is the probability to find the particle at this position.



## 3.2 Impuls

The next fundamental physical observable is impulse.

It is well known from the classical mechanics that impulse conservation law is closely related to the homogeneity of space i.e. the fact that physical laws are the same independent of where we put origin of our coordinate systems. More generally: each symmetry results in conserved property and the conserved property corresponding to the space translation symmetry is called 'impulse'.

Consider eigenvector  $|\xi_0\rangle$  of the  $x$ -coordinate which corresponds to the particular eigenvalue  $x = 0$  and let us assume that there is an operator  $\hat{U}(dx)$  which shifts the reference frame by to  $dx$  to the left. To say this is the same as to say, that the  $\hat{X}$ -eigenvector  $|\xi_0\rangle$  with the eigenvalue  $x = 0$  will be transformed to the vector  $|\xi_{dx}\rangle$  with the eigenvalue  $x = dx$ :

$$\hat{U}(dx)|\xi_0\rangle = |\xi_{dx}\rangle \quad (3.3)$$

If the operator depends 'continuously' on its argument, one can write  $\hat{U}(dx) = \hat{1} + \alpha \hat{P} dx$ , where  $\hat{P}$  is a new operator called 'generator' and  $\alpha$  is a convenience coefficient which should be considered as part of generator's definition. In the quantum theory  $\alpha$  is typically defined to be  $-\frac{i}{\hbar}$ . In analogy with the classic theory the generator of spatial translation is an operator corresponding to impulse observable.

Note that symmetry operators are 'unitary' operators characterized by the property  $\hat{U}^\dagger = \hat{U}^{-1}$  because we expect the scalar products (and thus experiment result probabilities) do not change under symmetry transformations - in fact this can be taken as definition of the term 'symmetry':

$$\langle \psi | \xi \rangle = \langle \hat{U} \psi | \hat{U} \xi \rangle = \langle \hat{U}^{-1} \hat{U} \psi | \xi \rangle. \quad (3.4)$$

As this must hold for all possible  $\xi$  and  $\psi$  the  $\hat{U}^\dagger = \hat{U}^{-1}$  follows. As result the generators of the symmetry are hermetian:

$$\hat{U}^\dagger = \hat{U}^{-1} = (\hat{1} + \alpha \hat{P} dx)^{-1} = \hat{1} - \alpha \hat{P} dx = \hat{1} + \frac{i}{\hbar} \hat{P} dx. \quad (3.5)$$

At the same time

$$\hat{U}^\dagger = \hat{1}^\dagger + \alpha^\dagger \hat{P}^\dagger dx^\dagger = \hat{1} + \frac{i}{\hbar} \hat{P}^\dagger dx, \quad (3.6)$$

which yields  $\hat{P}^\dagger = \hat{P}$ . It follows

$$\begin{aligned} dx &= dx \langle \xi_{dx} | \xi_{dx} \rangle = \langle \xi_{dx} | dx | \xi_{dx} \rangle = \langle \xi_{dx} | \hat{X} | \xi_{dx} \rangle = \\ &= \langle \hat{U}(dx) \xi_0 | \hat{X} | \hat{U}(dx) \xi_0 \rangle = \langle \xi_0 | \hat{U}^{-1}(dx) \hat{X} \hat{U}(dx) | \xi_0 \rangle = \\ &= \langle \xi_0 | (\hat{1} - \alpha dx \hat{P}) \hat{X} (\hat{1} + \alpha dx \hat{P}) | \xi_0 \rangle = \langle \xi_0 | \hat{X} | \xi_0 \rangle + \alpha dx \langle \xi_0 | [\hat{X}, \hat{P}] | \xi_0 \rangle = \\ &= \alpha dx \langle \xi_0 | [\hat{X}, \hat{P}] | \xi_0 \rangle, \end{aligned} \quad (3.7)$$

where the commutator  $[\hat{X}, \hat{P}]$  is defined as  $\hat{X} \hat{P} - \hat{P} \hat{X}$ . As this should hold for all  $dx$  it follows that

$$[\hat{X}, \hat{P}] = \frac{1}{\alpha} = i\hbar \quad (3.8)$$

Given that the operator  $\hat{X}$  acts by multiplying state-vector (function) with  $x$  one can see from (3.8) that

$$\hat{P} = -i\hbar \frac{\partial}{\partial x} \quad (3.9)$$

The eigenvector to the eigenvalue  $p$  is then the flat wave  $f_p(x) = e^{-\frac{i}{\hbar}px}$  which satisfy the equation:

$$\hat{P}f_p(x) = -i\hbar\frac{\partial f_p(x)}{\partial x} = pf_p(x) \quad (3.10)$$

For commutator  $[\hat{X}, \hat{P}] \equiv \hat{X}\hat{P} - \hat{P}\hat{X}$  we get

$$[\hat{X}, \hat{P}] = \hbar. \quad (3.11)$$

### 3.3 Energy

Energy is related to the time translations exactly as impulse to the spatial translation.

We assume that there is an operator  $\hat{U}(dt)$  which translate the state vector from time  $t = 0$  to  $t = dt$ , where  $dt$  is an infinitesimal small period of time. Again we write  $\hat{U}(dt) = \hat{1} - \frac{i}{\hbar}dt\hat{E}$ . If we want to translate to the arbitrarily amount of time  $T$  we could split it into small amounts  $T/N$  and make them infinitesimal by sending the natural number  $N$  to infinity. Now we must apply the operator  $\hat{U}(T/N)$  to the initial state vector  $N$  times to move it to the time  $t = T$ .

This yields the movement equation

$$|\xi(T)\rangle = \lim_{N \rightarrow \infty} \hat{U}^N(T/N)|\xi(0)\rangle = \lim_{N \rightarrow \infty} \left( \hat{1} - \frac{iT}{\hbar N} \hat{E} \right)^N |\xi(0)\rangle = e^{-\frac{iT}{\hbar} \hat{E}} |\xi(0)\rangle, \quad (3.12)$$

where exponent is formally defined as power series. Differentiating this by  $T$  we obtain:

$$\frac{\partial}{\partial T} |\xi(T)\rangle = -\frac{i}{\hbar} \hat{E} e^{-\frac{iT}{\hbar} \hat{E}} |\xi(0)\rangle = -\frac{i}{\hbar} \hat{E} |\xi(T)\rangle \quad (3.13)$$

or renaming for convenience  $T$  to  $t$ :

$$\frac{\partial}{\partial t} |\xi(t)\rangle = -\frac{i}{\hbar} \hat{E} |\xi(t)\rangle, \quad (3.14)$$

which is the famous Schrodinger's equation. It shows how the state evolves in time assuming the energy operator is known. As long we do not care about concrete form of the energy operator we can use (3.14) to formally introduce an energy operator:

$$\hat{E} = i\hbar \frac{\partial}{\partial t}. \quad (3.15)$$

For commutator  $[\hat{t}, \hat{E}] \equiv \hat{t}\hat{E} - \hat{E}\hat{t}$  we get

$$[\hat{t}, \hat{E}] = \hbar. \quad (3.16)$$

### 3.4 Uncertainty principle.

Let us define an uncertainty of the measurement of the observable  $A$  performed on the system in state  $|\Phi\rangle$  as follows:

$$\Delta_{\Phi}A = \langle (A - \langle A \rangle_{\Phi})^2 \rangle_{\Phi} \quad (3.17)$$

We follow [3] and express the product of uncertainties for two observables  $A$  and  $B$  through expectation value of the commutator of their operators  $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ . For this we use the following notation:

$$\Phi_A \equiv (\hat{A} - \langle A \rangle_\Phi) \Phi / \sqrt{\langle \Phi | \Phi \rangle} \quad (3.18)$$

$$\Delta_\Phi A = \sqrt{\langle \Phi_A | \Phi_A \rangle}. \quad (3.19)$$

From Schwarz inequality (2.11) we follow for any two operators  $\hat{A}$  and  $\hat{B}$ :

$$\Delta_\Phi A \Delta_\Phi B \geq |\langle \Phi_A | \Phi_B \rangle|. \quad (3.20)$$

At the same time

$$\langle \Phi_A | \Phi_B \rangle = \frac{\langle \Phi | [\hat{A} - \langle A \rangle_\Phi] [\hat{B} - \langle B \rangle_\Phi] \Phi \rangle}{\langle \Phi | \Phi \rangle} = \frac{\langle \Phi | [\hat{A}\hat{B} - \langle A \rangle_\Phi \langle B \rangle_\Phi] \Phi \rangle}{\langle \Phi | \Phi \rangle}. \quad (3.21)$$

For hermetian operator we can write  $\langle \Phi | \hat{A}\hat{B}\Phi \rangle = \langle \Phi | \hat{B}\hat{A}\Phi \rangle^*$  and, thus,

$$Im \langle \Phi_A | \Phi_B \rangle = \frac{\langle \Phi | [\hat{A}, \hat{B}] \Phi \rangle}{2i \langle \Phi | \Phi \rangle} = \langle [\hat{A}, \hat{B}] \rangle_\Phi / 2i \quad (3.22)$$

or

$$\Delta_\Phi A \Delta_\Phi B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle_\Phi|. \quad (3.23)$$

In particular for (3.11) and (3.16) we have the famous Heisenberg uncertainty relations:

$$\Delta_\Phi P \Delta_\Phi X \geq \frac{\hbar}{2} \quad (3.24)$$

and

$$\Delta_\Phi E \Delta_\Phi t \geq \frac{\hbar}{2}. \quad (3.25)$$

### 3.5 Schrodinger's equation for wave function

In mechanics the energy  $E$  consists of the kinetic part  $p^2/2m$  and of the potential part which is just a function of coordinate  $V(x)$ :

$$E = \frac{p^2}{2m} + V(x). \quad (3.26)$$

Before we can put this into (3.14) we must switch to operator notation. According to procedure known as 'first quantization', we replace the observables (impulse, coordinates etc.) with their operators. We know already how to write both: the impulse operator (3.9) and the coordinate operator, which is just a multiplication with this coordinate:

$$\begin{aligned}\hat{E} &= \frac{\hat{P}^2}{2m} + \hat{V}(x) \\ i\hbar \frac{\partial}{\partial t} &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x).\end{aligned}\tag{3.27}$$

This operator equation ultimately makes sense only once we apply it to vector state:

$$i\hbar \frac{\partial}{\partial t} |\xi\rangle = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] |\xi\rangle.\tag{3.28}$$

This is a differential equation for state vector  $|\xi(x, t)\rangle = \xi(x, t)$  which is, according to our choice, just a function of coordinate  $x$ . The operator on the right side is called Hamiltonian and denoted as  $\hat{H}$ :

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x).\tag{3.29}$$

As discussed above, for the particle in the state  $|\xi(x, t)\rangle = \xi(x, t)$  the probability to find the particle in the position  $x = x_0$  at time  $t = t_0$  equals to the square module of projection on the eigenvector  $\delta(x_0)$ :

$$P(x_0, t_0) = |\langle \xi(x, t_0) | \delta(x_0) \rangle|^2 = \left| \int_{-\infty}^{+\infty} \xi(x, t_0)^* \delta(x_0) dx \right|^2 = |\xi(x_0, t_0)|^2.\tag{3.30}$$

The state vector function  $\xi(x, t)$  is called 'one-particle wave function' and its evolution is governed by the Schrodinger's equation (3.28).

## 3.6 Historical considerations

Historically the motivation for the wave equation (3.28) was less axiomatic. Early in 20-th century Plank in his work about radiation of the black body and Einstein (in his work about photo effect) have proposed the following relationships between energy and impulse of the photon:

$$\begin{aligned}E &= \hbar\omega, \\ p &= \hbar k.\end{aligned}\tag{3.31}$$

In early 20-er years de Broglie postulated that all particles (not just photons) share these wave properties with electromagnetic waves. Let us assume that the free particle can somehow be described by a plane wave:

$$\xi(x, t) = e^{i\alpha(kx - \omega t)}\tag{3.32}$$

In mechanics the relationship between energy and impulse of the particle is like (3.26). In particular for the free particle  $V(x) = 0$ :

$$E = \frac{p^2}{2m}. \quad (3.33)$$

How can (3.33) be linked with (3.31)?

Let us look closer on how energy and impuls operators (3.9) and (3.15) act on the plane wave (3.32):

$$\begin{aligned} \hat{P}\xi(x, t) &= -i\hbar \frac{\partial}{\partial x} e^{i\alpha(kx - \omega t)} = \hbar k \xi(x, t) \\ \hat{E}\xi(x, t) &= i\hbar \frac{\partial}{\partial t} e^{i\alpha(kx - \omega t)} = \hbar \omega \xi(x, t) \end{aligned} \quad (3.34)$$

it looks encouraging similar to (3.31) and motivates the 'first quantization' procedure: 'take the energy-impulse relationship of classic mechanics (something like (3.33)) and replace energy and impulse with the operators'.

This procedure is called sometimes Correspondence Principle: an observable, which can be classically expressed as a function of impuls and coordinates, should be replaced by a quantum mechanical operator built by substituting the impuls by the impuls operator (3.9) and substituting the coordinates trivially by corresponding operator. The only complication arise from the fact that operators like  $\hat{P}\hat{x}$  are not Hermetian. To solve this the Correspondence Principle prescribes to replace them by hermetian  $(px + xp)/2$ .

Historically the energy as function of coordinates and impulse and energy operator are called Hamiltonian and symbols  $H$  or  $\hat{H}$  are used respectively.

# Chapter 4

## Composed systems.

Consider two systems. One described by the vectors from the vector space  $X$  and another described by the vectors from the vector space  $Y$ . Now consider the composed system, which consists of both. What could be the vector space  $Z$  of the possible states of the composed system?

The first idea would be take cartesian product (also called the direct sum) of both vector spaces:  $Z = X \times Y$ . This vector space consists of all pairs  $(x, y)$  with  $x \in X$  and  $y \in Y$ . The basis of it is just a union of the basis of  $X$  and the basis of  $Y$  and the dimension is the sum of dimensions of the both subspaces. Pairs  $(x, y)$  describe the states of both subsystems just by listing the state of the first system and the state of the second system one after another. This looks like description of two independent (not interacting) systems which one decided to describe together. However, the interaction between the subsystems can result in completely new states of the composed system, which are not just a combination of independent states for  $X$  and  $Y$ . This would require a vector space  $Z$  bigger than  $X \times Y$ .

Now, the quantum mechanics is a linear theory: the superposition of state vectors is linear, the operators for observables are linear, the time evolution equation is linear. Any operator, which acts linearly on the state vectors of each subsystem (such operator is called bilinear) should also act linearly on the state of combined system. In other words, any bilinear map  $b$  defined on  $X \times Y$  should correspond to some linear map  $b'$  defined on  $Z$ . This also implies some bilinear map  $a$  between  $X \times Y$  and  $Z$ , so that the following diagram commutes:

$$\begin{array}{ccc} X \times Y & \xrightarrow{a} & Z \\ & \searrow b & \downarrow b' \\ & & T \end{array}$$

The word 'commutes' mean  $b'(a(x, y)) = b(x, y)$  for all pairs  $(x, y) \in X \times Y$ . In algebra the vector space  $Z$  satisfying (uniquely) these requirements is called tensor product  $X \otimes Y$ . One can prove that the basis of  $X \otimes Y$  consists of all pairs  $(x_i, y_j)$  of the basis vectors  $x_i$  and  $y_j$  of  $X$  and  $Y$  respectively. We conclude that the state space of the composed system is the tensor product of the state spaces of its subsystems.

### 4.1 Entanglement.

Note that the tensor product introduced in the previous section is potentially much bigger space than  $X \times Y$ . In particular the dimension of the tensor product space is a product of dimensions of its factors. One can define an injection

$$\begin{aligned} X \times Y &\rightarrow X \otimes Y \\ (x, y) &\mapsto x \otimes y \end{aligned}$$

For example the tensor product of  $v = x_1 - 2x_2 \in X$  and  $w = y_1 + y_2 \in Y$  would be  $v \otimes w = x_1 \otimes y_1 - 2x_2 \otimes y_1 + x_1 \otimes y_2 - 2x_2 \otimes y_2$ , but one can easily construct a vector in  $X \otimes Y$  which cannot be written as product of a vector from  $X$  and a vector from  $Y$ : for example  $x_1 \otimes y_1 + x_2 \otimes y_2$ . Such states, which cannot be written as a product, are called entangled. Many phenomena in quantum informatics are related to entanglement.

Consider a quantum system which has only two states. Such systems are called quantum bits or qbits. The state of  $N$  of such qbits is described by a vector in the tensor product of  $N$  copies of the 2-dimensional  $\mathbb{C}$ -vector space. In other words the state is an element in  $\mathbb{C}^{2^N}$  - the  $2^N$  -dimensional  $\mathbb{C}$ -vector space. Compare this with the classical situation: the state of  $N$  classical bits is an element of a set of just  $2^N$  elements in total.

The existence of entangled states for composed system, which can not be written a tensor product of states of subcomponents leads to an interesting observation: assume the system is in the entangled state  $x_1 \otimes y_1 + x_2 \otimes y_2$  and assume further that someone performs measurement on the first system and finds the value corresponding to the basis eigenvector  $x_1$ . In this case, assuming the projective measurement, the wave function of the composed system collapses to  $x_1 \otimes y_1$  and the measurement on the second subsystem will certainly produce value corresponding to the eigenvector  $y_1$ . Assume further that the subsystems are already far away from each other (we leave for now by side an interesting question why and how this is possible). In this case the measurement performed on the first subsystem instantly has produced change in probability of measurement results of the distant second subsystem. In physics these considerations are known as paradox of Einstein, Podolsky and Rosen (EPR) and were used by authors in 1935 to show that quantum mechanics eventually contradict the idea of finite maximal speed of information transmission. We will revisit this once we have quantitative definition of information but let us now evaluate few simple cases.

First of all, assume that both observers, one possessing the first subsystem and another possessing the second one, did not agree any communication protocol in advance. The only common knowledge they have is the knowledge about the state of the composed system

$$|\Phi\rangle = \sum_{i,j} \alpha_{ij} |x_i\rangle \otimes |y_j\rangle. \quad (4.1)$$

As the quantum theory allows only statistical predictions, we may give to our observers many identical copies of the system, all in the same state, so allow them to make as many measures as they need to calculate the probability of different results they get. Would it be possible for the second observer to figure out which results the first observer gets and whether he performed any observation at all?

Let's assume that the first observer will perform a measurement and get the value corresponding to  $|x_n\rangle$ . The probability of this is  $P_{n*} = \sum_j |\alpha_{nj}|^2$  and the normalized state after the first measurement is

$$|\Phi_n\rangle = \frac{1}{\sqrt{\sum_j |\alpha_{nj}|^2}} |x_n\rangle \otimes \sum_j \alpha_{nj} |y_j\rangle. \quad (4.2)$$

The probability for the second observer to measure the value corresponding to  $|y_m\rangle$  under these circumstances is

$$P_{nm} = \frac{|\alpha_{nm}|^2}{\sum_j |\alpha_{nj}|^2}. \quad (4.3)$$

And the total probability to observe the value corresponding to  $|y_m\rangle$  is given by the weighted sum

$$P_{*m} = \sum_n P_{nm} P_{n*} = \sum_n \frac{|\alpha_{nm}|^2}{\sum_j |\alpha_{nj}|^2} \sum_j |\alpha_{nj}|^2 = \sum_n |\alpha_{nm}|^2. \quad (4.4)$$

Note that the same probability would be in case the first observer not even try to perform his measurement.

So, without protocol agreed in advance, it seems to be not very informative for the second observer to perform his measures and to count probabilities of different outcomes: he cannot say anything about the measurement done by the first observer. May be the first observer can change this by performing some different measurement? The most general attempt would be to measure arbitrary observable with the orthonormal eigenvectors

$$|\Psi^{(r)}\rangle = \sum_i \beta_i^{(r)} |x_i\rangle \quad (4.5)$$

with

$$\sum_i \beta_i^{(r)} \beta_i^s = \delta^{rs}. \quad (4.6)$$

Let's rewrite the initial state  $\Phi$  in this new basis:

$$|\Phi\rangle = \sum_r |\Psi^{(r)}\rangle (\sum_j \gamma_{rj} |y_j\rangle), \quad (4.7)$$

where coefficients  $\gamma$  can be found by comparing (4.7) with  $\alpha$  in (4.1):

$$|\Phi\rangle = \sum_{i,j} \alpha_{ij} |x_i\rangle \otimes |y_j\rangle = \sum_{i,j} (\sum_r \beta_i^{(r)} \gamma_{rj}) |x_i\rangle \otimes |y_j\rangle \quad (4.8)$$

and thus

$$\alpha_{ij} = \sum_r \beta_i^{(r)} \gamma_{rj}. \quad (4.9)$$

This system of linear equations can be solved using the orthonormality of  $\Psi^{(r)}$  vectors (4.6) by multiplying on both sides by  $\beta_i^s$  and summing over  $i$ :

$$\sum_i \beta_i^s \alpha_{ij} = \sum_r \gamma_{rj} \sum_i \beta_i^{(r)} \beta_i^s = \gamma_{sj}. \quad (4.10)$$

Thus the initial state  $\Phi$  can be rewritten as

$$|\Phi\rangle = \sum_r |\Psi^{(r)}\rangle \otimes \sum_j (\sum_i \beta_i^{(r)} \alpha_{ij}) |y_j\rangle \quad (4.11)$$



and the probability for the first observer to observe the value corresponding to  $|\Psi^{(r)}\rangle$  is

$$P_{r*} = \sum_j \left| \sum_i \beta_i^{(r)} \alpha_{ij} \right|^2. \quad (4.12)$$

According to projection rule, the state of the composed system after the first measurement will be

$$|\Phi^{(r)}\rangle = \frac{1}{\sqrt{\Omega^{(r)}}} |\Psi^{(r)}\rangle \otimes \sum_j \left( \sum_i \beta_i^{(r)} \alpha_{ij} \right) |y_j\rangle, \quad (4.13)$$

where  $\Omega$  is a normalization factor

$$\Omega^{(r)} = \sum_j \left| \sum_i \beta_i^{(r)} \alpha_{ij} \right|^2 = P_{r*}. \quad (4.14)$$

Under these circumstances, the probability  $P_{rm}$  that the second observer will measure the value corresponding to  $|y_m\rangle$ .

$$P_{rm} = \frac{\left| \sum_i \beta_i^{(r)} \alpha_{ij} \right|^2}{\sqrt{\Omega^{(r)}}}. \quad (4.15)$$

And the total probability to observe the value corresponding to  $|y_m\rangle$  is given again by the weighted sum

$$P_{*m} = \sum_r P_{rm} P_{r*} = \sum_r \frac{\left| \sum_i \beta_i^{(r)} \alpha_{ij} \right|^2}{\sqrt{\Omega^{(r)}}} \sum_j \left| \sum_i \beta_i^{(r)} \alpha_{ij} \right|^2 = \sum_r \left| \sum_i \beta_i^{(r)} \alpha_{ij} \right|^2 = \sum_r |\alpha_{rm}|^2, \quad (4.16)$$

where we used (4.6) on the last step.

Note that again this probability is the same as it would be if the first observer would perform any other measurement or not measure at all. What can the first observer do to affect the results of the second observer?

It could let his part of the system evolve according to Schrodinger equation with some unitary operator  $\hat{U}$ . The state (4.11) then would be

$$|\Phi\rangle = \sum_r \hat{U} |\Psi^{(r)}\rangle \otimes \sum_j \left( \sum_i \beta_i^{(r)} \alpha_{ij} \right) |y_j\rangle, \quad (4.17)$$

but this just redefines the vectors  $\Psi^{(r)}$  leaving them orthonormal with  $\langle \Psi^{(r)} | \hat{U}^* \hat{U} | \Psi^{(r)} \rangle$  and, thus, does not change the situation.

The described situation is sometimes referred as **no-communication theorem**. Note that quantum mechanics, which would allow non-linear evolution, could violate this theorem. Also interpretation of the state vector other than as a simple vehicle for calculation of the probabilities for observed values, could break the arguments stated above: the state vector of the composed system is clearly changed by the measurement performed by the remote observer, but, as long we use the state vector for calculation of probabilities only, the result of this calculation is not affected by the remote measurement.

However, as discussed above, one can compose entangled state which would allow the second observer to know the outcome of the experiment made by the first observer instantly by performing a measurement on

his part of the system. Knowing, that the first observer already finished his measurement on the composed system, which was initially prepared in the state  $(x_1 \otimes y_1 + x_2 \otimes y_2)/\sqrt{2}$ , the second observer will with certainty know the result of the first experiment: for example, if performing his own measurement the second observer will get the value corresponding to  $|y_1\rangle$  he will know that the first observer got the value corresponding to  $|x_1\rangle$ , because otherwise the state of the composed system after the first measurement would be  $x_2 \otimes y_2$  leaving no chance to observe the value corresponding to  $|y_1\rangle$ .

This looks promising: the second observer can make statements about results of the first observer with delay less than the time the light would need to travel between both! Does this contradict the relativity theory? We promised to revisit this, once we introduce a quantitative measure for information. Intriguing is a superficial similarity between the described scenario and the following communication protocol: our two observers agree to buy the newspaper in advance but to look at the title page first at midnight. In this case the second observer would also immediately and with certainty know which title story the first observer is reading. With some cheat he could even predict it in advance.

## 4.2 Open systems

Consider a system composed of two qubits and assume that some physical property  $A$  belongs to the first qubit. It could be spin or impulse or position in space. In the vector space of the first qubit, considered alone, we would have a hermitian operator  $\hat{A}$  for this observable  $A$ . Being an observable of the first qubit, the observable  $A$  is also an observable of the system composed of two qubits. Which operator corresponds to  $\hat{A}$  in the tensor product space of the two qubits? The answer is  $\hat{A} \otimes \hat{1}$ , where product operator acts on the basis vectors as follows:  $\hat{A} \otimes \hat{B}(e_i \otimes h_j) = \hat{A}e_i \otimes \hat{B}h_j$ .

Assume that the composed system is in the state  $|\phi\rangle$  and we want to calculate an expectation value of the operator  $\hat{A}' = \hat{A} \otimes \hat{1}$ . We can use the density operator of the composed system to calculate it and hope that the density operator in the product space would allow similar decomposition as  $\hat{A}'$  or at least some simplification, if we restrict the application to the observables of the first qubit only.

Let the initial state of the composed system be

$$|\phi\rangle = \sum_{i,j} a_{ij} |e_i h_j\rangle, \quad (4.18)$$

where we write  $|e_i \otimes h_j\rangle$  as  $|e_i h_j\rangle$  for convenience. As a basis  $\{e_k\}$  for the first qubit we choose the orthonormal eigenvectors of the operator  $\hat{A}$ :  $\hat{A}e_k = \lambda_k e_k$ .

The density operator for the composed system is per definition

$$\hat{\rho} = |\phi\rangle\langle\phi| = \sum_{i,j,m,n} a_{ij} a_{mn}^* |e_i h_j\rangle\langle e_m h_n|, \quad (4.19)$$

and the expectation value for  $\hat{A}'$  is

$$\begin{aligned}
\langle A' \rangle &= Tr(\hat{\rho}\hat{A}') = \sum_{k,l} \langle e_k h_l | \sum_{i,j,m,n} a_{ij} a_{mn}^* | e_i h_j \rangle \langle e_m h_n | \hat{A} \otimes \hat{\mathbb{1}} | e_k h_l \rangle \\
&= \sum_{k,l} \langle e_k h_l | \sum_{i,j} a_{ij} | e_i h_j \rangle \sum_{mn} a_{mn}^* \langle e_m h_n | \hat{A} \otimes \hat{\mathbb{1}} | e_k h_l \rangle \\
&= \sum_{k,l} a_{kl} | \sum_{mn} a_{mn}^* \langle e_m h_n | \hat{A} \otimes \hat{\mathbb{1}} | e_k h_l \rangle \\
&= \sum_{kl} |a_{kl}|^2 \lambda_k, \tag{4.20}
\end{aligned}$$

where we used  $\hat{A}e_k = \lambda_k e_k$  at the last step.

This can also be expressed in terms of the reduced density operator  $\hat{\rho}_1 = Tr_2(\hat{\rho}) \equiv \sum_{im} \left[ \sum_j a_{ij} a_{mj}^* \right] |e_i\rangle\langle e_m|$ , where  $Tr_2$  means a partial trace (the sum over the index  $j$ ) over vector space dimensions of the second qubit only.

Here is the proof:

$$\begin{aligned}
Tr(\hat{\rho}_1 \hat{A}) &= \sum_k \langle e_k | \hat{\rho}_1 \hat{A} | e_k \rangle = \sum_k \langle e_k | \sum_{im} \left[ \sum_j a_{ij} a_{mj}^* \right] |e_i\rangle\langle e_m| \hat{A} | e_k \rangle \\
&= \sum_{kl} |a_{kl}|^2 \lambda_k \tag{4.21}
\end{aligned}$$

comparing (4.20) with (4.21) we conclude, that

$$\langle A' \rangle = Tr(\hat{\rho}\hat{A}') = Tr(\hat{\rho}_1 \hat{A}) = \langle A \rangle, \tag{4.22}$$

where  $\hat{A}' = \hat{A} \otimes \hat{\mathbb{1}}$ .

This is a nice result. It allows us to use the density operator  $\hat{\rho}_1$ , defined in the vector space of the first qubit only, instead of  $\hat{\rho}$ , defined in the bigger vector space of the composed system, as long we are dealing with the first qubit only, even if both are entangled.

### 4.3 Isolation and interaction

Per definition, the state of a composed system is not entangled, if it can be written as a tensor product of a vector for the first component and a vector for the second component. Can the system, starting with such non entangled state  $|\phi\rangle \otimes |\xi\rangle$ , evolve into an entangled state without interaction between components?

The answer is no. Here is the argument.

Without an interaction, the energy of the composed system should be a sum of the energies of its parts. Writing the non-entangled eigenvector of the hamiltonian of the composed system as  $|\phi\rangle \otimes |\xi\rangle$ , one would expect

$$\hat{H}|\phi\rangle \otimes |\xi\rangle = (E_1 + E_2)|\phi\rangle \otimes |\xi\rangle, \tag{4.23}$$

which can be easily achieved with the hamiltonian of the form  $\hat{H} = \hat{H}_1 \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{H}_2$ , with

$$\hat{H}_1|\phi\rangle = E_1|\phi\rangle, \quad (4.24)$$

$$\hat{H}_2|\xi\rangle = E_2|\xi\rangle. \quad (4.25)$$

Both parts of the hamiltonian obviously commute  $[\hat{H}_1 \otimes \hat{1}, \hat{1} \otimes \hat{H}_2] = 0$ . For commuting operators  $\hat{A}$  and  $\hat{B}$ , an exponent of a sum is a product of exponents:  $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}$  and, thus, an initially non-entangled state  $|\phi\rangle \otimes |\xi\rangle(0)$  evolves with the time as follows:

$$\begin{aligned} |\phi\rangle \otimes |\xi\rangle(t) &= e^{-\frac{it}{\hbar}[\hat{H}_1 \otimes \hat{1} + \hat{1} \otimes \hat{H}_2]}|\phi\rangle \otimes |\xi\rangle(0) \\ &= e^{-\frac{it}{\hbar}\hat{H}_1 \otimes \hat{1}}e^{-\frac{it}{\hbar}\hat{1} \otimes \hat{H}_2}|\phi\rangle \otimes |\xi\rangle(0) = e^{-\frac{it}{\hbar}\hat{H}_1}|\phi\rangle(0) \otimes e^{-\frac{it}{\hbar}\hat{H}_2}|\xi\rangle(0), \end{aligned} \quad (4.26)$$

with the last expression clearly representing an non-entangled state vector. Given that the Schrodinger equation supports reverse of time, the opposite statement is also true: existing entanglement will not dissappear without interaction between the entangled subsystems.

# Chapter 5

## Application to informatics.

### 5.1 No clone.

### 5.2 Teleportation

### 5.3 Superdense coding

Consider one possible orthonormal basis of the state space of two qubits - the 4-dimntional  $\mathbb{C}^2 \otimes \mathbb{C}^2$  vector space:

$$\begin{aligned} |\Phi_+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ |\Phi_-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\Psi_+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |\Psi_-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned} \tag{5.1}$$

These states are called Bell states.

Assume that Alyse possess the first and Bob the second qubit and the whole system is initially in one of these basis sates, say in  $|\Phi_+\rangle$ . This state is entangeled and Alyse cannot remove entangelement by acting on her qubit locally by means of non-interavting evolution operator  $\hat{U}(t) = e^{-\frac{i\hat{t}}{\hbar}[\hat{H}_1 \otimes \hat{1} + \hat{1} \otimes \hat{H}_2]}$ . But she can tranform it to any other base vector she wants, assuming that Bob does not chnage state of his qubit.

It may be worth to explain, what do we mean by non-changing state. Without interaction, the state of the second qubit will evolve by operator  $e^{-\frac{i\hat{t}}{\hbar}\hat{H}_2}$ . If both vectors  $|0\rangle$  and  $|1\rangle$  are the eigenvectors of  $\hat{H}_2$  with the same eigenvalue  $E_2$ , the evolution of both vectors yields the same common factor  $e^{-\frac{i\hat{t}}{\hbar}E_2}$ , which does not change probabilities of any measurement. We expect, that Bob choose the Hamiltonian  $\hat{H}_2$  carefully to satisfy this requirement. For exmpale, in case the vectors  $|0\rangle$  and  $|1\rangle$  are characterized by different magnetic moments, Bob should not expose his qubit to an external magnetic field.

Now Alyse can prepare her qubit so, that the composed system evolves into any of the four Bell states (5.1). For exmpale, by choosing an appropriate  $\hat{H}_1$ , she can can make  $e^{-\frac{i\hat{t}}{\hbar}\hat{H}_1 \otimes \hat{1}} = \hat{Z} \otimes \hat{1}$  with  $Z_{01} = Z_{10} = 1$

and  $Z_{00} = Z_{11} = 0$ . This evolution would switch the vectors  $|0\rangle$  and  $|1\rangle$  of the first qubit and so transform  $|\Phi_+\rangle$  into  $|\Psi_+\rangle$ .

In this way Alyse may associate four different messages with four different Bell states and thus code two bits of information into a system composed of two entangled qubits. This is not surprising. But if Alyse next sends her qubit to Bob, who will now possess the complete composed system, she will put him into position to perform an experiment and distinguish the four orthogonal Bell vectors and the message associated with them. Note that Alyse has sent two bits of information to Bob by transferring only one qubit. This procedure is called superdense coding.

The preparation of the entangled initial state of two qubits is done in advance, before the act of communication and can be considered as a process of establishing a communication channel. Once the channel is in place, Alyse is free to decide, which one message out of four she wants to transfer to Bob. During the communication process itself, only one qubit is transferred.

## 5.4 Deutsch Jozsa algorithm.

## 5.5 Fourier transformation.

## 5.6 Phase estimation.

## 5.7 Factoring.

# Chapter 6

## Hardware.

### 6.1 Harmonic oscillator.

Hamiltonian of the harmonic oscillator has the following form:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2. \quad (6.1)$$

Instead of solving Schrodinger's equation directly one can introduce operators

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{q} + i\hat{p}) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{q} - i\hat{p}) \end{aligned} \quad (6.2)$$

with commutator  $[\hat{a}, \hat{a}^\dagger] = 1$ . The Hamiltonian (6.1) can then be written as

$$\hat{H} = \frac{1}{2} \hbar\omega(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \quad (6.3)$$

Consider operator  $\hat{N} = \hat{a}^\dagger\hat{a}$ . Assuming it has an eigenvector  $|\Psi\rangle$ , the corresponding eigenvalue  $\lambda$  must be positive:

$$\langle\Psi|\hat{N}|\Psi\rangle = \lambda\langle\Psi|\Psi\rangle = \langle\Psi|\hat{a}^\dagger\hat{a}|\Psi\rangle = \langle\hat{a}\Psi|\hat{a}\Psi\rangle \geq 0. \quad (6.4)$$

Operators  $\hat{a}$  and  $\hat{a}^\dagger$  create a new eigenvector of the operator  $N$  with the eigenvalue  $\lambda - 1$  and  $\lambda + 1$  correspondingly. One can introduce now the eigenvector  $|0\rangle$  with the lowest possible eigenvalue  $\lambda_0$  and using notation  $|n\rangle$  for the eigenvector with the eigenvalue  $\lambda_0 + n$  write

$$\begin{aligned} \hat{a}|0\rangle &= 0, \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle, \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle, \\ |n\rangle &= \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle. \end{aligned} \quad (6.5)$$

These factors  $\sqrt{n}$  and  $\sqrt{n+1}$  result from the normalization requirement  $\langle n|n\rangle = 1$ .

Energy spectrum of the oscillator is then given by

$$H|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle. \quad (6.6)$$

This is the famous Plank formula which he introduced 1900 as an assumption.

For the sake of completeness we note, that the solution of the Schrodinger's equation (3.28) with hameltonian (6.1) is given by so called Hermite polynomials, named after the French mathematician Charles Hermite, of the hannonical form:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \quad (6.7)$$

The eigenvectors of Hamiltonian are called sometimes 'stable' because they evolve with time by phase factor only. According to the Schroedinger equation the time evolution is given by the unitary operator  $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$ :

$$|n(t)\rangle = e^{-i\hat{H}t/\hbar}|n\rangle = e^{-in\omega t}|n\rangle, \quad (6.8)$$

where we ignored a constant phase  $i\hbar\omega/2$ , which is the same for all  $n$ .

How one would implement quantum computer with the oscillator? Let's consider two qubits and CNOT gate as trivial sample program. The first idea would be to code  $|00\rangle_q$  as  $|0\rangle_o$ ,  $|01\rangle_q$  as  $|1\rangle_o$ ,  $|10\rangle_q$  as  $|2\rangle_o$  and  $|11\rangle_q$  as  $|3\rangle_o$ , where we use the subscript  $q$  to indicate a two-qubits state and the the subscript  $o$  to indicate an oscillator state.

However, if we try to implement CNOT with time evaluation operator (6.8), we will find another choice more convinient:

$$\begin{aligned} |00\rangle_q &\rightarrow |0\rangle_o, \\ |01\rangle_q &\rightarrow |1\rangle_o, \\ |10\rangle_q &\rightarrow (|4\rangle_o + |1\rangle_o)/\sqrt{2}, \\ |11\rangle_q &\rightarrow (|4\rangle_o - |1\rangle_o)/\sqrt{2}. \end{aligned} \quad (6.9)$$

Now, for  $t = \pi/\omega$  the  $|n\rangle_o$  does not change for even  $n$  and switch the sign for odd under (6.8). In other words, under  $\hat{U}(\pi/\omega)$  the  $|01\rangle_q$  and  $|00\rangle_q$  are not changed but  $|10\rangle_q$  evolves to  $|11\rangle_q$  and vs. versa. This is exactly what one would expect from CNOT gate.

Note that the coding table (6.9) does not scale well: for  $m$  qubits coded together one would need energy of  $2^m \times \hbar\omega$ , which is a lot compared with  $m \times \hbar\omega$  in case, one would use the  $m$  qubits separately.



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